9.2 (continued)

last time: $f(t)$ period $2L$, given on $-L<t<L$

$$f(t)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi t}{L}\right)+b_n\sin\left(\frac{n\pi t}{L}\right)\right]$$

$$a_n=\frac{1}{L}\int_{-L}^{L}f(t)\cos\left(\frac{n\pi t}{L}\right)dt \qquad n=0,1,2,3,\dots$$

$$b_n=\frac{1}{L}\int_{-L}^{L}f(t)\sin\left(\frac{n\pi t}{L}\right)dt \qquad n=1,2,3,\dots$$

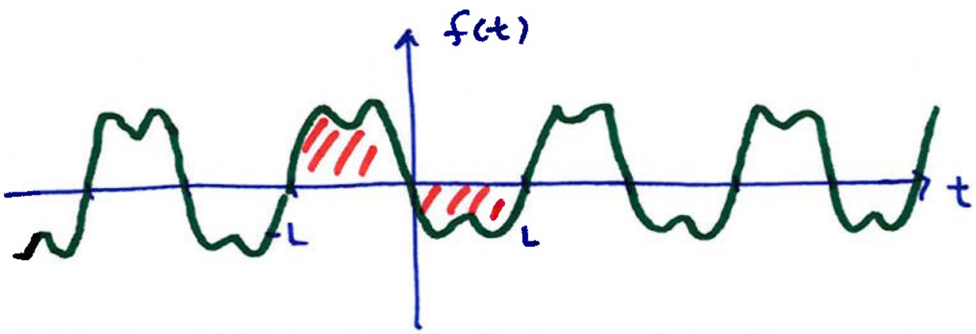
in practice, $-L<t<L$ is not always convenient

often we want $0<t<2L$

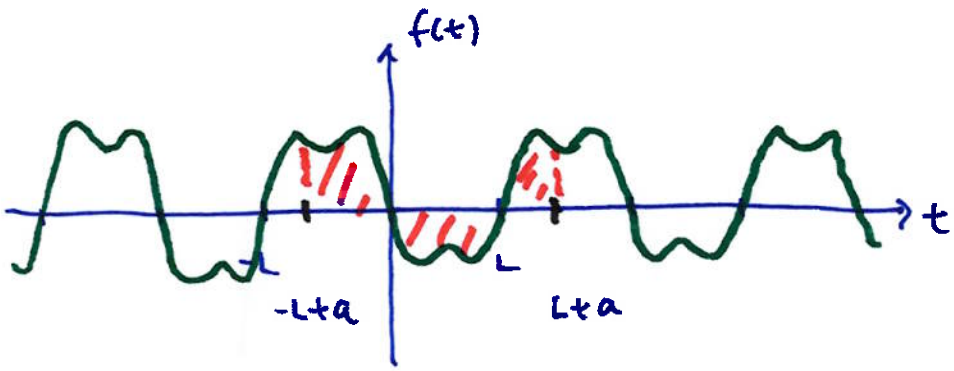
how do the formulas above change?

$$f(t)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi t}{L}\right)+b_n\sin\left(\frac{n\pi t}{L}\right)\right]$$ stays the same

$g(t)$ has period $2L$, it should look like

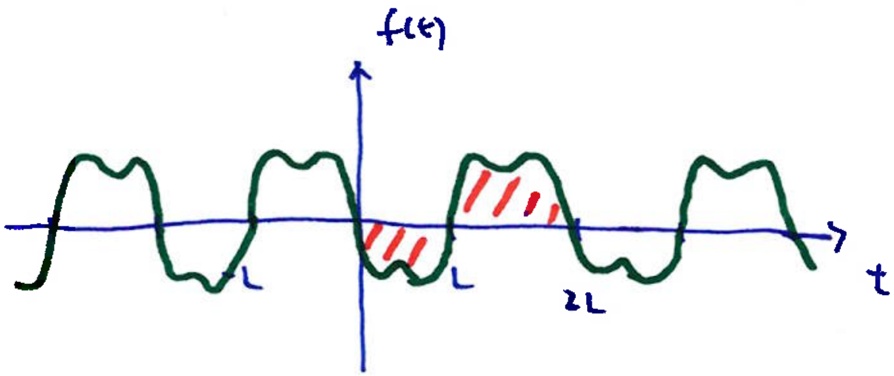


$\int_{-L}^{L} g(t)\,dt$ is the shaded area



$\int_{-L+a}^{L+a} g(t)\,dt$ is the shaded area

notice it's the same



$\int_{0}^{2L} g(t)\,dt$ is still the same!

we see $\displaystyle\int_{-L}^{L} g(t)\,dt = \int_{-L+a}^{L+a} g(t)\,dt = \int_{0}^{2L} g(t)\,dt$

must have period 2L

let's look at the coefficient formulas

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(t)\cos\left(\frac{n\pi t}{L}\right)}_{\text{must have period } 2L} dt$$

$f(t)$ has period $2L$ by definition

$\cos\left(\frac{n\pi t}{L}\right)$ has period $\dfrac{2\pi}{n\pi/L} = \dfrac{2L}{n}$ fundamental period

but any integer multiple is also a period

so, $\cos\left(\frac{n\pi t}{L}\right)$ also has period of $2L$

and $f(t)\cos\left(\frac{n\pi t}{L}\right)$ also has period of $2L$

so, we can shift the integration interval however we want

as long as its length is 2L (period)

more natural Fourier series formulas:

f(t) period 2L

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right]$$

$$a_n = \frac{1}{L}\int_0^{2L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

$$b_n = \frac{1}{L}\int_0^{2L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

---

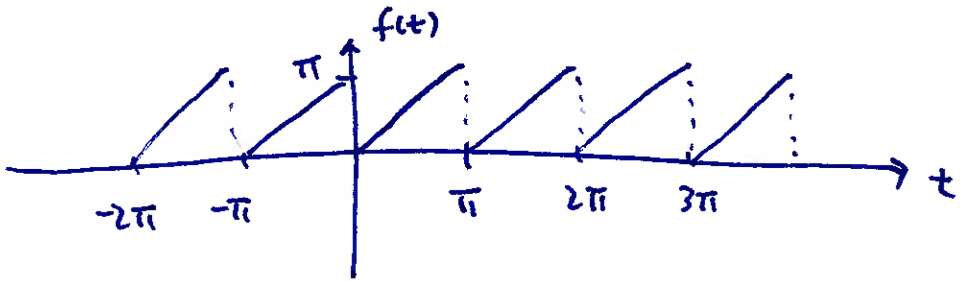
or, since period $p = 2L$

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2n\pi t}{p}\right) + b_n \sin\left(\frac{2n\pi t}{p}\right)\right]$$

$$a_n = \frac{2}{p}\int_0^{p} f(t)\cos\left(\frac{2n\pi t}{p}\right)dt$$

$$b_n = \frac{2}{p}\int_0^{p} f(t)\sin\left(\frac{2n\pi t}{p}\right)dt$$

example $\quad f(t) = t \quad 0 < t < \pi \quad$ period $\pi \quad (L = \frac{\pi}{2})$



$$a_0 = \frac{1}{\pi/2} \underbrace{\int_0^{\pi} t \, dt}_{\text{area}} = \frac{2}{\pi}\left(\frac{1}{2}\cdot\pi\cdot\pi\right) = \pi$$

$$\frac{n\pi t}{L} = \frac{n\pi t}{\pi/2}$$

$$a_n = \frac{1}{\pi/2} \int_0^{\pi} t \cos(2nt)\, dt$$

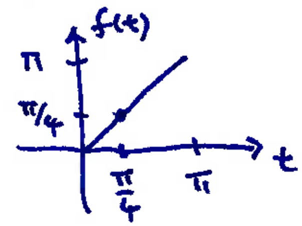
$$= \cdots = 0$$

$$b_n = \frac{1}{\pi/2} \int_0^{\pi} t \sin(2nt)\, dt = \cdots = -\frac{1}{n}$$

$$f(t) \sim \frac{\pi}{2} + \sum_{n=1}^{\infty} -\frac{1}{n} \sin(2nt)$$

$$t = \frac{\pi}{2} - \sin(2t) - \frac{1}{2}\sin(4t) - \frac{1}{3}\sin(6t) - \frac{1}{4}\sin(8t) - \cdots$$

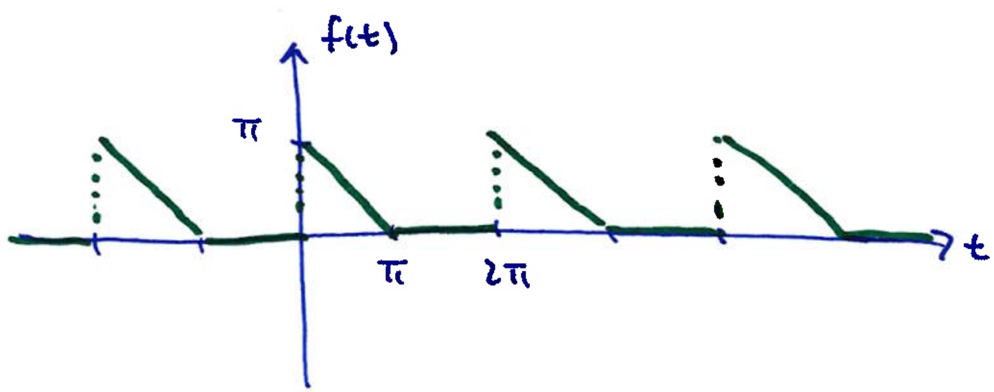
evaluate at $t = \frac{\pi}{4}$



$$\frac{\pi}{4} = \frac{\pi}{2} - 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \cdots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots \quad \text{Leibniz series}$$

<u>example</u> $f(t) = \begin{cases} \pi - t & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{cases}$ period 2π $(L = \pi)$



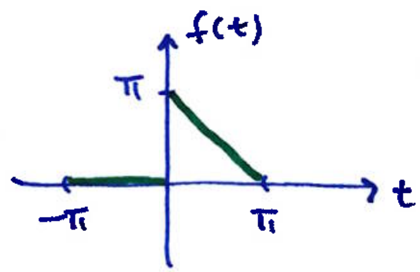
$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(t)\,dt = \frac{1}{\pi}\left(\frac{1}{2}\cdot\pi\cdot\pi\right) = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)\,dt = \cdots = \frac{1-(-1)^n}{n^2\pi} = \begin{cases} 0 & n \text{ is even} \\ \frac{2}{n^2\pi} & n \text{ is odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)\,dt = \cdots = \frac{1}{n}$$

$$f(t) \sim \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2\pi}\cos(nt) + \frac{1}{n}\sin(nt)\right]$$

let's evaluate at $t = 0$



function is discontinuous at $t = 0$

Fourier series goes to $\dfrac{f(0^-)+f(0^+)}{2} = \dfrac{\pi}{2}$

$$\frac{\pi}{2} = \frac{\pi}{4} + \sum_{n \text{ odd}}^{\infty} \frac{2}{n^2\pi}\overbrace{\cos(nt)}^{1}$$

$$= \frac{\pi}{4} + \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right)$$

simplify

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots$$

Sum of reciprocals of odd squares

on HW, you are asked to show $\frac{\pi^2}{6} = \cancel{1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} +} \cdots$ (Basel problem)

$$= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$